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Algebraic Determination of the Metric from the Curvature in General Relativity

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The general solution for a symmetric second-order tensor X of the equation $X_{e(a}R^{e}{}_{b)cd}=0$, where R is the Riemann tensor of a space-time manifold, and X is obtained in terms of the curvature 2-form structure of R by a straightforward geometrical technique, and agrees with that given by McIntosh and Halford using a different procedure. Two results of earlier authors are derived as simple corollaries of the general theorem.

1. INTRODUCTION

Let M be a space-time, a four-dimensional manifold carrying a Lorentz metric g of signature $+2$ and let $p\in M$. McIntosh and Halford (1981, 1982) have discussed the equation

$$X_{e(a}R^{e}{}_{b)cd}=0 \tag{1}$$

at p, where the X_{ab} and $R^{a}{}_{bcd}$ are the components of a symmetric second-order tensor and the Riemann tensor, respectively, in some coordinate system about p. Their aim was to study the types of Riemann tensor for which (1) has solutions for X_{ab} other than the trivial solution $X_{ab}=\phi g_{ab}$, where the g_{ab} are the components of the metric tensor at p and $\phi\in\mathbb{R}$. The equation (1) has arisen in at least two recent discussions: (i) the algebraic determination of the metric tensor components from a given set of Riemann tensor components (Ihrig 1975a,b; 1976; McIntosh and Halford, 1981,

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1982). This problem arises naturally in the application of the equation of geodesic deviation to the scattering of a cloud of neutral test particles (Pirani, 1956; Szekeres, 1965), (ii) the discussion of curvature collineations. A curvature collineation is a vector field ξ defined on some open subset U of M such that $\mathcal{L}_\xi R^a{}_{bcd}=0$ holds in U, where $\mathcal{L}_\xi$ denotes the Lie derivative along the paths of ξ. A necessary condition for the existence of a curvature collineation ξ on U is that (1) holds in U with $X_{ab}=\xi_{(a;b)}$ (Katzin et al., 1969). Thus the existence of a nontrivial solution of (1) is a necessary condition for the existence of a curvature collineation which is not a conformal motion since for the latter, $\xi_{(a;b)}$ is proportional to g_{ab} (Collinson, 1970; McIntosh and Halford, 1982).

McIntosh and Halford (1982) investigated the problem of nontrivial solutions of (1) by considering the canonical forms for a symmetric second-order tensor given by Plebański (1964). It turned out that for nontrivial solutions of (1) to exist, the maximum dimension of the bivector space spanned by the curvature 2-forms was 3, thus strengthening a result due to Ihrig (1975a). In this paper, a direct proof of this result will be given which is shorter and more amenable to a geometrical interpretation. It should be stressed that the approach given will be a purely algebraic study of (1) at the point p and no differentiability requirements will be taken into account.[1]

2. THE MAIN RESULTS

At $p\in M$, one introduces a real null tetrad of vectors l, n, y, z, where l and n are null vectors and y and z unit spacelike vectors and where the only nonvanishing inner products between the tetrad members are $y^a y_a = z^a z_a = -l^a n_a = 1$. The metric tensor at p is related to the tetrad vectors by the completeness relation $g_{ab}=-2l_{(a}n_{b)}+y_a y_b+z_a z_b$. With this tetrad one can construct the associated 1-forms at p

$$\theta^1 = l_a dx^a, \qquad \theta^2 = n_a dx^a, \qquad \theta^3 = y_a dx^a, \qquad \theta^4 = z_a dx^a \tag{2}$$

written in terms of some coordinate system (x^a) about p. The corresponding curvature 2-forms at p are then given by

$$\theta^a{}_b = \tfrac{1}{2} R^a{}_{bcd}\theta^c \wedge \theta^d \tag{3}$$

Here the $R^a{}_{bcd}$ are the components of the curvature tensor in the basis l, n, y, z.

[1]Some results concerning the differentiability requirements have been given by Collinson and da Graça Lopes Rodrigues Vaz (1982), and by Hall (1982).

One can now establish two results which show how the algebraic structure of each of the curvature 2-forms at p imposes certain restrictions on the algebraic structure of the symmetric tensor X_{ab} through the equation (1).

Theorem 1. If at $p \in M$, F is a simple 2-form (bivector) whose blade is spanned by the vectors r and s and if X is a symmetric second-order tensor, then the following two conditions are equivalent:

(i) $X_{c(a}F^{c}{}_{b)} = 0$.

(ii) The vectors r and s are eigenvectors of X with equal eigenvalues.

Proof. In components, one has $F_{ab} = r_{[a}s_{b]}$ and

$$X_{c(a}F^{c}{}_{b)} = 0 \Leftrightarrow r^{c}X_{c(a}s_{b)} = s^{c}X_{c(a}r_{b)}$$

$$\Leftrightarrow X_{ab}r^{b} = \alpha r_{a} \quad \text{and} \quad X_{ab}s^{b} = \alpha s_{a} \qquad (\alpha \in \mathbb{R}) \qquad \blacksquare$$

Now suppose that F is a nonnull 2-form at p. If l and n generate the unique pair of null eigendirections of F and if x and y are orthogonal spacelike vectors spanning the spacelike 2-space orthogonal to that spanned by l and n, then with an appropriate scaling, one may construct a real null tetrad l, n, y, z at p where the tetrad vectors satisfy the conditions given earlier. It then follows that there exists $\alpha, \beta \in \mathbb{R}$ such that $F_{ab} = \alpha l_{[a}n_{b]} + \beta y_{[a}z_{b]}$. Further, if F is nonsimple then α and β are both nonzero.

Theorem 2. If at $p \in M$, F is a nonsimple 2-form and X a symmetric second-order tensor then with the notation established above the following two conditions are equivalent:

(i) $X_{c(a}F^{c}{}_{b)} = 0$.

(ii) The null vectors l and n are eigenvectors of X with equal eigenvalues and the spacelike vectors y and z are eigenvectors of X with equal eigenvalues.

Proof. Suppose that condition (i) holds and write $F_{ab} = \alpha l_{[a}n_{b]} + \beta y_{[a}z_{b]}$ where α and β are both nonzero (since F is not simple). Then on contracting (i) successively with l^{b}, n^{b}, y^{b}, and z^{b} one finds

$$X_{cb}l^{b}F^{c}{}_{a} = \alpha X_{ac}l^{c} \tag{4}$$

$$X_{cb}n^{b}F^{c}{}_{a} = -\alpha X_{ac}n^{c} \tag{5}$$

$$X_{cb}y^{b}F^{c}{}_{a} = \beta X_{ac}z^{c} \tag{6}$$

$$X_{cb}z^{b}F^{c}{}_{a} = -\beta X_{ac}y^{c} \tag{7}$$

Next, since $\beta \neq 0$, it follows easily that l and n are the only eigendirections of F and that they satisfy $l^a F_{ab} = \alpha l_b$ and $n^a F_{ab} = -\alpha n_b$. Hence it follows from (4) and (5) that there exist real numbers γ and δ such that $X_{ab} l^b = \gamma l_a$ and $X_{ab} n^b = \delta n_a$, the symmetry of X showing that $\gamma = \delta$. Finally, since l and n span a timelike invariant 2-space of X at p, the spacelike 2-space orthogonal to it is also an invariant 2-space of X at p and consequently contains an orthogonal pair of spacelike eigenvectors of X (Churchill, 1932; Hall, 1976; 1979). Without loss in generality these eigenvectors will be supposed to coincide with the vectors y and z above. Equation (6) or (7) then shows that y and z have equal eigenvalues and statement (ii) in the theorem follows. The converse is immediately proved by applying Theorem 1 to the 2-forms $l_{[a}n_{b]}$ and $y_{[a}z_{b]}$. ■

If the conditions and statements in Theorem 2 hold then it easily follows that

$$X_{ab} = -2\gamma l_{(a}n_{b)} + \mu(y_a y_b + z_a z_b) \qquad (\gamma, \mu \in \mathbb{R}) \tag{8}$$

Thus the only eigenvectors admitted by X lie either in the 2-space spanned by l and n or that spanned by y and z, unless $\gamma = \mu$, in which case the completeness relation shows that $X_{ab} = \gamma g_{ab}$. It follows that if this trivial solution is *not* to be the only solution of the equation $X_{c(a}F^c{}_{b)} = 0$, the only 2-forms which may satisfy this equation must be linear combinations of $l_{[a}n_{b]}$ and $y_{[a}z_{b]}$. This is a consequence of the previous two theorems since any other 2-forms satisfying this equation would give rise to eigenvectors of X outside the blades of the 2-forms $l_{[a}n_{b]}$ and $y_{[a}z_{b]}$.

On applying these results to the equation (1) one sees that if nontrivial solutions for X are required and if the Riemann tensor has nonsimple curvature 2-forms, then these curvature 2-forms span a subspace of the space of all 2-forms at p of dimension at most 2. If the dimension is 2, then this subspace will be spanned by the simple 2-forms corresponding to $l_{[a}n_{b]}$ and $y_{[a}z_{b]}$ above. The possibility of this subspace having dimension 1 leads to a contradiction since it implies the existence of a nonsimple 2-form N satisfying $R_{abcd} = \pm N_{ab}N_{cd}$. But then the requirement $R_{a[bcd]} = 0$ implies that $N_{a[b}N_{cd]} = 0$, which is equivalent to N being simple. The conclusion is that if (1) is to have nontrivial solutions for X, then the curvature 2-forms must span a subspace of the space of all 2-forms at p for which a basis of simple 2-forms may be chosen. This greatly simplifies the procedure and the situation concerning (1) can be summarized in the next theorem.

Theorem 3. If equation (1) holds at p, then at p

(i) if the curvature 2-forms are spanned by a single (necessarily simple) 2-form F, then

$$X_{ab} = \phi g_{ab} + \mu u_a u_b + 2\nu u_{(a}v_{b)} + \lambda v_a v_b \tag{9}$$

where $\phi, \mu, \nu, \lambda \in \mathbb{R}$ and u and v span the 2-space orthogonal to the blade of F;

(ii) if the curvature 2-forms span a subspace of bivector space of dimension 2 or 3 and if the members of this subspace have a common eigenvector w with zero eigenvalue (necessarily unique up to a real scaling factor), then

$$X_{ab} = \phi g_{ab} + \lambda w_a w_b \tag{10}$$

where $\phi, \lambda \in \mathbb{R}$;

(iii) if the curvature 2-forms are spanned by the 2-forms $l_{[a}n_{b]}$ and $y_{[a}z_{b]}$ where l, n, y, z constitute a real null tetrad at p, then

$$X_{ab} = \phi g_{ab} + 2\lambda l_{(a}n_{b)} = (\phi - \lambda)g_{ab} + \lambda(y_a y_b + z_a z_b) \tag{11}$$

where $\phi, \lambda \in \mathbb{R}$;

(iv) in all other cases

$$X_{ab} = \phi g_{ab} \tag{12}$$

where $\phi \in \mathbb{R}$, is the only solution.

Proof. Equation (1) imposes restrictions of the form $X_{e(a}F^e{}_{b)} = 0$ on X for each curvature 2-form F. So if in part (i) of the theorem F has a spacelike blade, say $F_{ab} = y_{[a}z_{b]}$ with y and z orthogonal spacelike vectors, then Theorem 1 shows that y and z are eigenvectors of X with equal eigenvalues. On completing y and z to a real null tetrad l, n, y, z and on considering X as a linear combination of symmetrical products of the tetrad members, one easily finds

$$\begin{aligned} X_{ab} &= \alpha_1 l_a l_b + \alpha_2 n_a n_b + 2\alpha_3 l_{(a}n_{b)} + \alpha(y_a y_b + z_a z_b) \\ &= \alpha g_{ab} + \alpha_1 l_a l_b + \alpha_2 n_a n_b + 2(\alpha + \alpha_3) l_{(a}n_{b)} \end{aligned} \tag{13}$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, with α the common eigenvalue of y and z and where the completeness relation has been used. This expression is of the required form since the 2-space spanned by l and n is orthogonal to that spanned by y and z. A similar calculation can be used to establish the relevant expression when the blade of F is timelike or null and also to establish the result in part (ii) of the theorem. Now suppose that the curvature 2-forms are spanned by two simple 2-forms whose blades intersect only in the zero vector (equivalently whose blades are such that there is no nonzero vector orthogonal to both of them). If these blades are spanned by the vectors u_1

and u_2 and by v_1 and v_2 then it follows that u_1, u_2, v_1, and v_2 are independent and Theorem 1 shows that they are eigenvectors of X where u_1 and u_2 have eigenvalue μ, say, and v_1 and v_2 have eigenvalue ν. If $\mu = \nu$, then since u_1, u_2, v_1, and v_2 constitute a basis for the tangent space $T_p(M)$ to M at p, it follows that $X_{ab} = \phi g_{ab}$ ($\phi \in \mathbb{R}$) is the only solution of (1). On the other hand, if $\mu \neq \nu$ then the 2-spaces spanned by u_1 and u_2 and by v_1 and v_2 must be orthogonal and so one must be timelike and one spacelike. Hence one may select a real null tetrad l, n, y, z such that one of these 2-spaces is spanned by l and n and the other by y and z. Part (iii) of the theorem now follows by considerations similar to those which led to equation (8). It now easily follows that if the curvature 2-forms are spanned either by three independent 2-forms such that there is no nonzero vector orthogonal to all of them, or by four or more independent 2-forms, then there are no nontrivial solutions of (1). For if there were, one could in both cases find a spanning set of simple 2-forms. In the former case either all members of this spanning set contain a certain vector or else they generate a nonsimple linear combination of themselves and neither of these possibilities leads to nontrivial solutions of (1). In the latter case, nonsimple linear combinations of the spanning 2-forms always occur and again the result follows. This completes the proof of part (iv) of the theorem. ■

3. DISCUSSION

Theorem 3 gives a complete description of equation (1) as regards the solutions for X given the curvature 2-form structure at p. It has a number of immediate consequences which can now be discussed.

(i) Theorem 3 gives a geometrical interpretation and a more systematic approach to some of the results given in McIntosh and Halford (1982), who essentially studied the problem in reverse by considering given canonical forms for X.

(ii) It follows as a consequence of Theorem 3 that if the curvature 2-forms span a vector space of dimension 6 at p, then $X_{ab} = \phi g_{ab}$ ($\phi \in \mathbb{R}$) is the only solution of equation (1). This result was first given by Ihrig (1975a).

(iii) For vacuum space-times, the algebraic structure of the Riemann tensor is conveniently given by the Petrov classification. In this scheme, the possible types are, in the usual notation, I, D, II, N, and III and their curvature 2-forms span subspaces of dimension 6, 6, 6, 2, and 4, respectively. Hence the only possibility for nontrivial solutions of (1) occurs at these points where the Riemann tensor is of Petrov type N. In this case, the canonical form for such a Riemann tensor shows that the curvature 2-forms are spanned by a null 2-form and its dual and it then follows from Theorem 3(ii) that the general solution of (1) is $X_{ab} = \phi g_{ab} + \psi l_a l_b$ ($\phi, \psi \in \mathbb{R}$), where

l^a is the null vector which spans the repeated principal null direction of the Riemann tensor (and its null curvature 2-forms). This recovers the result first given by Collinson (1970).

(iv) In those cases in Theorem 3 where the trivial solution $X_{ab}=\phi g_{ab}$ is not the only solution of (1) one can readily evaluate the Segré type, or, equivalently, the Plebański type of the solutions for X (Plebański, 1964; Hall, 1976; 1979). In fact, in case (i) of Theorem 3 when F has a spacelike blade, equation (13) shows that if exactly one of α_1 and α_2 is nonzero, the Segré type of X is $\{2(1,1)\}$ (Plebański type $[2N-2S]_{(2-1)}$) or some degeneracy of this type. If, however, $\alpha_1=\alpha_2=0$ then the Segré type is $\{(1,1)(1,1)\}([2T-2S]_{(1-1)})$ or some degeneracy of this type. If α_1 and α_2 are both nonzero, then the null tetrad may be adjusted so that $|\alpha_1|=|\alpha_2|$. The two cases $\alpha_1=\alpha_2$ and $\alpha_1=-\alpha_2$ then lead, respectively, to the Segré types $\{1,1(1,1)\}([T-S_1-2S_2]_{(1-1-1)})$ and $\{z,\bar{z}(1,1)\}([Z-\bar{Z}-2S]_{(1-1-1)})$ together with their possible degeneracies. If F has a timelike blade, a similar calculation shows that the only possible Segré type is $\{(1,1)1,1\}([2T-S_1-S_2]_{(1-1-1)})$ or one of its degeneracies. If F has a null blade then the only possibilities are the Segré types $\{(1,1,1)1\}([3T-S]_{(1-1)})$, $\{(2,1)1\}([3N-S]_{(2-1)})$ and $\{(3,1)\}([4N]_3)$ or their degeneracies. In case (ii) of Theorem 3, equation (10) shows that the possible Segré types are $\{1,(1,1,1)\}([T-3S]_{(1-1)})$, $\{(1,1,1)1\}([3T-S]_{(1-1)})$ or $\{(2,1,1)\}([4N]_{(2)})$, together with their possible degeneracies, according as w^a is timelike, spacelike, or null. In case (iii) of Theorem 3, the only possible Segré type is $\{(1,1)(1,1)\}([2T-2S]_{(1-1)})$ or its degeneracy. These results agree with those in Table 1 of McIntosh and Halford (1982).

(v) The statement that, at p, the curvature 2-forms span a one-dimensional subspace of the space of all 2-forms [Theorem 3(i)] is equivalent to the statement that there are exactly two independent nonzero vectors k satisfying

$$R_{abcd}k^d=0 \tag{14}$$

This follows since (14) is equivalent to the vector k being orthogonal to all of the curvature 2-forms at p. In the notation of Theorem 3(i), the solutions k of (14) span a 2-space orthogonal to the blade of F. Similarly one shows that the situation in Theorem 3(ii) is equivalent to the existence of a single independent nonzero solution k of (14). In the notation of Theorem 3(ii) the solution of (14) (up to a real multiple) is the vector w. Only when the situation is as described in Theorem 3(iii) are there nontrivial solutions X of (1) without the existence of nonzero solutions k of (14). A more detailed account of equation (14) has been given by McIntosh and Van Leeuwen (1982).

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